

Born–Infeld theory of gravitation: Spherically symmetric static solutions

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1 Introduction

In the work [2] a theory of electroweak and gravitational fields based on the Born–Infeld type of action was suggested. In this paper the attention is narrowed to the gravitational sector only; respectively, all fields, except gravitational, and matter are considered to be absent. In the third section the modified vacuum Einstein equations are derived from the Born–Infeld action. In the fourth section the equations for the static spherically symmetric case are considered in a more detail. The asymptotics for the Schwarzschild solution as a decomposition in parameter $L = (20/3)^{\frac{1}{2}}(k^{\frac{1}{2}}\hbar)/(ec) \approx 10^{-32}$ cm is obtained. In the fifth section an interior solution is obtained, also with static, spherically symmetric spacial geometry corresponding to the region $r \in [0, r_0)$ with $r_0 \propto L$.

2 Motivation and basic notions

More than sixty years ago Born and Infeld suggested a theory [1] of electromagnetism with non-quadratic lagrangian. The action of fields,

$$S_{BI} = \int d\Omega \left(\sqrt{-\det |\eta_{ab} + f_{ab}|} - \sqrt{-\det |\eta_{ab}|} \right), \quad (2.1)$$

leads to non-linear Maxwell equations for dimensionless tensor of electromagnetic field, f_{ab} . Here tensor η_{ab} represents flat spacetime metric. Authors were inspired by a certain result of their theory, namely, by absence of the singularity in a special case of static, spherically symmetric electric field. One may use the same approach, treating gravitation. As in the case of electromagnetic fields, one may expect that non-linear (in curvature tensor) corrections to the Einstein equations should cancel with linear term on small distances in such manner, that singularity (in the case of the Schwarzschild solution) disappears. Accepting the idea of non-linear Einstein equations (in a sense given above), one has a lot of alternatives for selecting different powers of, say, the Riemann tensor (contracted with

metrics) with arbitrary coefficients. So, one appears to be helpless to select the unique lagrangian. In this respect Born–Infeld approach suggests a method for construction of (almost) unique lagrangian, by using the only dimensional parameter, which happens to be a characteristic length, $L \approx 10^{-32}$ cm.

One can further decrease the number of competing theories, by demanding that the theory should be compatible with quantum field theory. This means the following. The Dirac action for a fermion (described by Dirac’s 4–spinor, ψ), doesn’t contain the metric tensor, g_{ab} , and the Christoffel symbol, Γ_{ab}^c , as fundamental elements. Instead, the basic notions, from which geometry can be derived, are the set of Dirac matrices, γ_a , satisfying relations,

$$\gamma_{(a} \gamma_{b)} = g_{ab} \hat{1}, \quad (2.2)$$

(here $\hat{1}$ is the unit matrix 4×4) and a set of spinorial connections, Γ_a , associated with a covariant derivative. Both sets are matrices 4×4 .

Accepting these notions as fundamental, one may construct theory as follows. Introduce dimensionless operators, π_a , according to formula,

$$\pi_a \Psi = -iL(\partial_a - \Gamma_a)\Psi. \quad (2.3)$$

Then, one may construct an operator,

$$\phi_{ab} = \gamma_{[a} \gamma_{b]} - \pi_{[a} \pi_{b]}. \quad (2.4)$$

The last term in (2.4) is proportional to the curvature tensor,

$$\rho_{ab} = 2\pi_{[a} \pi_{b]} \quad (2.5)$$

$$= L^2 (\partial_a \Gamma_b - \partial_b \Gamma_a - [\Gamma_a, \Gamma_b]). \quad (2.6)$$

Next, one may construct a scalar density (c -number), using only operator ϕ_{ab} :

$$\phi = \frac{1}{4!} e^{abcd} e^{efgh} \frac{1}{4} \text{Tr} [\phi_{ae} \phi_{bf} \phi_{cg} \phi_{dh}]; \quad (2.7)$$

here the absolute antisymmetric symbol $e^{abcd} = e^{[abcd]} = \pm 1$. For the action of gravitational field in absence of other interactions (i.e. of electroweak and strong) one may take the following expression:

$$S_g = K \int d\Omega [\sqrt{-\phi} - \sqrt{5}\sqrt{-g}]. \quad (2.8)$$

Here $g = \det \left[\frac{1}{4} \text{Tr} (\gamma_a \gamma_b) \right]$. It’s worthy to mention here, that first term in the action (2.8) (i.e. $\sqrt{-\phi}$) is form-invariant with respect to transformations,

$$\gamma_a = \cosh \theta \gamma'_a + \sinh \theta \pi'_a; \quad (2.9)$$

$$\pi_a = \sinh \theta \gamma'_a + \cosh \theta \pi'_a; \quad (2.10)$$

here θ is some constant. One may claim this symmetry as fundamental and demand all terms in the action to be invariant with respect to (2.9) and (2.10). A theory of electroweak and gravitational fields, based on this idea, is constructed in [2]; the reader is referred to this work for details. As it is shown, the characteristic length, $L = (20/3)^{\frac{1}{2}}(k^{\frac{1}{2}}\hbar)/(ec)$.¹ In absence of other fields and matter, the action suggested in [2] is reduced to (2.8).

3 Vacuum equations for gravitational field

Taking variation of (2.8) in γ_a and Γ_a , one obtains the following set of equations:

$$\frac{1}{\sqrt{-g}} \partial_a [\sqrt{-g} \phi^{ba}] - [\Gamma_a, \phi^{ba}] = 0; \quad (3.1)$$

$$[\phi^{ab}, \gamma_a] = 2\gamma^b. \quad (3.2)$$

Here $\gamma^b = g^{ba}\gamma_a$ and g^{ab} is a contravariant metric tensor ($g_{ab} = \frac{1}{4} \text{Tr}(\gamma_a \gamma_b)$); the following definitions are used:

$$\phi^{dh} = \frac{\sqrt{-\phi}}{\sqrt{-5g}} \varphi^{dh}; \quad (3.3)$$

$$\varphi^{dh} = \frac{1}{3!\phi} e^{abcd} e^{efgh} \phi_{ae} \phi_{bf} \phi_{cg}. \quad (3.4)$$

In [2] it is shown that in the limit $L \rightarrow 0$ action (2.8) (and hence equations (3.1), (3.2)) are equivalent to those of Einstein's theory.

4 Spherically symmetric static metrics

To test equations (3.1), (3.2), one may apply them to the case of static, spherically symmetric gravitational field. The difference between usual approach and the one employed here is the spinorial representation of basic notions (i.e. of Dirac matrices, γ_a and connections, Γ_a). To begin with, one assumes that each component of ϕ^{ab} , defined by (3.3), is proportional to $\gamma^{[a} \gamma^{b]}$. (Later it will be shown that this assumption really takes place.) Then, from (3.2) follows,

$$\phi^{01} = -\frac{m}{2+m} \gamma^{[0} \gamma^{1]}; \quad (4.1)$$

$$\phi^{23} = -\frac{m}{2+m} \gamma^{[2} \gamma^{3]}; \quad (4.2)$$

¹This formula was obtained by taking into consideration only of electroweak and gravitational fields; introduction of the strong interaction should change the numeric value. The order of magnitude, though, shouldn't change substantially.

$$\phi^{0\sigma} = -\frac{1}{2+m} \gamma^{[0} \gamma^{\sigma]}; \quad \sigma = 2, 3; \quad (4.3)$$

$$\phi^{1\sigma} = -\frac{1}{2+m} \gamma^{[1} \gamma^{\sigma]}. \quad (4.4)$$

Here m is some function. From (3.3), (4.1) – (4.4), one obtains,

$$\phi_{01} = \alpha_1 \gamma_{[0} \gamma_{1]}; \quad (4.5)$$

$$\phi_{23} = \alpha_1 \gamma_{[2} \gamma_{3]}; \quad (4.6)$$

$$\phi_{0\sigma} = \alpha_2 \gamma_{[0} \gamma_{\sigma]}; \quad (4.7)$$

$$\phi_{1\sigma} = \alpha_2 \gamma_{[1} \gamma_{\sigma]}. \quad (4.8)$$

Take for the interval usual expression, namely,

$$ds^2 = e^{2\lambda} dt^2 - e^{2\nu} dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\varphi^2]. \quad (4.9)$$

Here λ and ν are functions of r only, and (t, r, θ, φ) are spherical coordinates, having traditional interpretation. Then, for Dirac matrices one can take

$$\gamma_0 = e^\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (4.10)$$

$$\gamma_1 = e^\nu \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}; \quad (4.11)$$

$$\gamma_2 = r \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}; \quad (4.12)$$

$$\gamma_3 = r \sin \theta \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}. \quad (4.13)$$

The following notations are used for linear combinations of standard Pauli matrices,

$$\sigma_1 = \begin{pmatrix} \cos \theta & \sin \theta \exp(-i\varphi) \\ \sin \theta \exp(i\varphi) & -\cos \theta \end{pmatrix}; \quad (4.14)$$

$$\sigma_2 = \begin{pmatrix} -\sin \theta & \cos \theta \exp(-i\varphi) \\ \cos \theta \exp(i\varphi) & \sin \theta \end{pmatrix}; \quad (4.15)$$

$$\sigma_3 = \begin{pmatrix} 0 & -i \exp(-i\varphi) \\ i \exp(i\varphi) & 0 \end{pmatrix}. \quad (4.16)$$

Using (3.1), (4.10) – (4.16), one obtains the following formulae for connections: $\Gamma_0 = a_0 e^{-\lambda-\nu} \gamma_{[0} \gamma_{1]}$; $\Gamma_1 = 0$; $\Gamma_\sigma = \frac{1}{r} a_2 e^{-\nu} \gamma_{[1} \gamma_{\sigma]}$; $\sigma = 2, 3$. Here one uses notations,

$$a_0 = \frac{1}{2} e^{\lambda-\nu} \left[\lambda' - \frac{m'(1+m)}{2+m} - \frac{m^2-1}{r} \right]; \quad (4.17)$$

$$a_2 = \frac{1}{2} - \frac{e^{-\nu}}{2} \left(m + \frac{rm'}{2+m} \right). \quad (4.18)$$

Here prime denotes differentiation over r . For the curvature tensor components one obtains expressions:

$$\rho_{01} = -e^{-\lambda-\nu} a_0' \gamma_{[0} \gamma_{1]}; \quad (4.19)$$

$$\rho_{0\sigma} = \frac{e^{-\lambda}}{r} a_0 (2a_2 - 1) \gamma_{[0} \gamma_{\sigma]}; \quad (4.20)$$

$$\rho_{1\sigma} = \frac{e^{-\nu}}{r} a_2' \gamma_{[1} \gamma_{\sigma]}; \quad (4.21)$$

$$\rho_{23} = \frac{2}{r^2} a_2 (1 - a_2) \gamma_{[2} \gamma_{3]}. \quad (4.22)$$

From (3.3), (4.1) – (4.8), one obtains algebraic relations between α_1 , α_2 and m :

$$\frac{\sqrt{15}m}{2+m} = \frac{\alpha_1(3\alpha_1^2 + 2\alpha_2^2)}{\sqrt{3\alpha_1^4 + 4\alpha_1^2\alpha_2^2 + 8\alpha_2^4}}; \quad (4.23)$$

$$\frac{\sqrt{15}}{2+m} = \frac{\alpha_2(\alpha_1^2 + 4\alpha_2^2)}{\sqrt{3\alpha_1^4 + 4\alpha_1^2\alpha_2^2 + 8\alpha_2^4}}. \quad (4.24)$$

Solving eqs. (4.23), (4.24) with respect to α_1 and α_2 , one obtains,

$$\alpha_1(m) = \frac{\sqrt{15}k}{2+m} \sqrt{\frac{mk+2}{k^2+4}}; \quad (4.25)$$

$$\alpha_2(m) = \frac{\sqrt{15}}{2+m} \sqrt{\frac{mk+2}{k^2+4}}; \quad (4.26)$$

here $k = \alpha_1/\alpha_2$ is a real root of equation,

$$k^3 - \frac{m}{3}k^2 + \frac{2}{3}k - \frac{4m}{3} = 0; \quad (4.27)$$

The root can be written explicitly as

$$k = \frac{1}{9}m + \frac{1}{3}A(m) + \frac{1}{3}B(m), \quad (4.28)$$

where

$$A(m) = \sqrt[3]{17m + \frac{1}{27}m^3 + \sqrt{8 + \frac{863}{3}m^2 + \frac{4}{3}m^4}}; \quad (4.29)$$

$$B(m) = \sqrt[3]{17m + \frac{1}{27}m^3 - \sqrt{8 + \frac{863}{3}m^2 + \frac{4}{3}m^4}}. \quad (4.30)$$

From (2.4), (4.5) – (4.8), (4.19) – (4.22), one obtains after some manipulation with formulae, expressions for λ and ν as functions of r , α_1 , α_2 , m :

$$e^\nu = \frac{L^2}{4r(\alpha_2 - 1)} \left[\sqrt{1 - \frac{4r^2}{L^2}(1 - \alpha_1)} \right]'; \quad (4.31)$$

$$\lambda' = \left[\frac{1}{2} \ln \left(1 - \frac{4r^2}{L^2}(1 - \alpha_1) \right) + m - \ln(2 + m) \right]' + \frac{m^2 - 1}{r}. \quad (4.32)$$

Further, there are two differential equations, imposed on functions in case:

$$\frac{r\alpha_1'}{2(\alpha_2 - 1)} - \frac{rm'}{2 + m} - m + \frac{\alpha_1 - 1}{\alpha_2 - 1} = 0; \quad (4.33)$$

$$\frac{r\alpha_2'}{(\alpha_2 - 1)} + \frac{rm'}{2 + m} + \left(\frac{rm'}{2 + m} + m \right) \left(m - \frac{\alpha_1 - 1}{\alpha_2 - 1} \right) = 0. \quad (4.34)$$

It is easy to show that compatibility condition for (4.33) and (4.34) can be written as

$$\dot{\alpha}_2 + \frac{m}{2}\dot{\alpha}_1 + \frac{\alpha_2 - \alpha_1}{2 + m} = 0, \quad (4.35)$$

where dot denotes differentiation over m . This condition actually holds provided that (4.25) – (4.27) take place; thus one may consider only eq. (4.33), disregarding (4.34).

Solving (4.33), one obtains,

$$r = r_\star \sqrt[3]{|2 + m|} |m(\alpha_2 - 1) - (\alpha_1 - 1)|^{-\frac{1}{2+m^2}} F(m), \quad (4.36)$$

where r_\star is a constant of integration, and

$$F(m) = \frac{1}{\sqrt[6]{2 + m^2}} \exp \left(-\frac{1}{3\sqrt{2}} \arctan \frac{m}{\sqrt{2}} \right) \exp \left[-\int_1^m dm' M(m') \right]; \quad (4.37)$$

$$M(m) = \frac{2m}{(2 + m^2)^2} \ln |m(\alpha_2 - 1) - (\alpha_1 - 1)|. \quad (4.38)$$

Inverting (4.36), one obtains function $m(r)$. Function $F(m)$ in (4.37) ‘behaves properly’, i.e. it doesn’t have zeroes and poles in finite range of m . In the case $m' \neq 0$, i.e. when

formula (4.36) takes place (case $m = \text{constant}$ will be considered separately), one may obtain the following formulae for metrics coefficients, using (4.31) – (4.33), (4.36),

$$g_{00} = \frac{r_*^6}{r^6} \left[1 + \frac{4r^2}{L^2}(\alpha_1 - 1) \right] \frac{(2+m)^2}{[m(\alpha_2 - 1) - (\alpha_1 - 1)]^2}; \quad (4.39)$$

$$g_{11} = - \left[1 + \frac{4r^2}{L^2}(\alpha_1 - 1) \right]^{-1} \frac{[m(2+m)\dot{\alpha}_1 - 2(\alpha_1 - 1)]^2}{[(2+m)\dot{\alpha}_1 - 2(\alpha_2 - 1)]^2}. \quad (4.40)$$

Equations (4.39), (4.40), (4.36) (together with respective definitions) give formal solution to the problem. One should bring to attention, though, the fact that function $r(m)$ is not monotonic; so one has to cut interval $(-\infty, +\infty) \ni m$ into domains of monotony of the function $r(m)$. Each such domain would correspond to some solution to the problem. One may distinguish the following intervals: (i) $(-\infty, -2)$; (ii) $(-2, m_0)$; (iii) (m_0, m_{min}) ; (iv) $(m_{min}, 1)$; (v) $(1, +\infty)$. Here $m_0 \approx -1.60808367$; $m_{min} \approx -1.365056$. Function $r(m)$ behaves monotonically in each interval. Consider each interval separately.

Case $m \in (1, +\infty)$: The Schwarzschild solution, negative mass

On interval function $r(m)$ decreases from $+\infty$ to a constant value. At the left border of interval $r \rightarrow \infty$, metrics corresponds to the Schwarzschild solution with $r_g < 0$.

Case $m \in (m_{min}, 1)$: The Schwarzschild solution, positive mass

Interval corresponds to an exterior solution. Function $r(m)$ increases from minimal value $r = \text{constant}$ to $+\infty$. At the right border of the interval ($m \rightarrow 1$) one may expand all functions in case into series in small parameter $(1 - m)$; inverting the expansion for $r(m)$, one may find $m(r)$. Omitting details of computation, one may present the following formulae for metrics' expansion in parameter L (up to terms of order L^4):

$$g_{00} = \left(1 + \frac{3r_*^6}{r^6} \right) \left[1 - \frac{r_g}{r} \left(1 - \frac{r_*^6}{r^6} \right) \right]; \quad (4.41)$$

$$g_{11} = - \left(1 - \frac{9r_*^6}{r^6} \right) \left[1 - \frac{r_g}{r} \left(1 - \frac{r_*^6}{r^6} \right) \right]^{-1}. \quad (4.42)$$

Here the following notation is used:

$$r_*^6 = \frac{r_g^2 L^4}{80}. \quad (4.43)$$

Here r_g is another constant, corresponding to the Schwarzschild radius of Einstein's theory. Constant r_* is connected to r_g by relation,

$$r_*^3 = p_1 r_g L^2, \quad (4.44)$$

where

$$p_1 = \frac{\sqrt{3}}{8} \exp \left(\frac{1}{\sqrt{2}} \arctan \frac{1}{\sqrt{2}} \right) \approx 0.3345645. \quad (4.45)$$

Due to (4.43) corrections of order L^2 are absent in metrics' decomposition. The domain of validity for (4.41), (4.42) is $r \gg r_*$. At the left border, $m = m_{min}$, $r(m)$ achieves minimum; one may expand, again, $r(m)$ into series near minimal value; thus,

$$\frac{r}{r_*} = A + B(m - m_{min})^2 + \dots \quad (4.46)$$

Numeric computations give $A \approx 0.48024254$, $B \approx 0.60170272$. Inverting (4.46) and substituting into (4.39), (4.40), one obtains the following asymptotics:

$$g_{00} = 15.144 \times f(m_{min}) + O\left(\frac{r}{r_{min}} - 1\right); \quad (4.47)$$

$$g_{11} = -0.495 \times [f(m_{min})]^{-1} \left(\frac{r}{r_{min}} - 1\right)^{-1} + O(1). \quad (4.48)$$

Here one uses the notation,

$$f(m) = 1 + \frac{4r^2}{L^2}(\alpha_1 - 1). \quad (4.49)$$

Value $r_{min} \equiv r(m_{min}) = Ar_*$ (c.f. (4.46)). Comparing this expression with (4.44), one obtains,

$$f(m_{min}) = 1 - q_0 \left(\frac{r_g}{L}\right)^{\frac{2}{3}}; \quad (4.50)$$

here $q_0 \approx 3.0767137$. Using numerical computations, one may show that $f(m)$ monotonically increases on interval $m \in (m_{min}, 1)$. One also has $f(1) = 1$. This implies that formation of a horizon (corresponding to $f(m_g) = 0$ for some $m_g \in (m_{min}, 1)$) depends on sign of $f(m_{min})$. Namely, horizon forms if and only if $f(m_{min}) < 0$. According to (4.50), masses $r_g < 0.185L$ don't form the horizon in the domain of validity, i.e. in the region $r > r_{min} \approx 0.333(r_g L^2)^{\frac{1}{3}}$. Note, that in this case r_{min} doesn't exceed $0.2L$. The coordinate system (t, r, θ, φ) cease to be valid in the region $r < r_{min}$. Discussion of some details of the solution will be postponed until the conclusion.

Case $m \in (m_0, m_{min})$: Interior solution I

Function $r(m)$ decreases from $r = +\infty$ to $r = \text{constant}$, achieving minimum. On the right border of interval formulae (4.47) and (4.48) are still valid. One has $\dot{f}(m) > 0$ (respectively, $f'(m) < 0$) on the interval. Besides, $f(m_0) = -\infty$. This means, that if $f(m_{min}) > 0$, then solution with correct signature ($g_{00} > 0$, $g_{11} < 0$) exists only for $r_{min} < r < r(m_u)$, where m_u is determined by equation $f(m_u) = 0$. Note, that since the Schwarzschild solution is not linked with the one in consideration, one cannot use formula (4.44); in this case r_* is independent constant.

Case $m \in (-2, m_0)$: Interior solution II

Function $r(m)$ increases from $r = 0$ to $r = +\infty$. At the left border of interval, asymptotics for the metric coefficients is the following:

$$g_{00} \propto \frac{r^2}{r_*^2}; \quad g_{11} \propto -\frac{r^4}{r_*^4}. \quad (4.51)$$

Here r_* is a constant of integration. On the interval $\dot{f}(m) < 0$; as it was mentioned above, $f(m_0) = -\infty$. One has a freedom to choose the constant of integration (r_* in (4.36)), so that $f(-2) > 0$. Thus, the solution exists for $0 < r < r_m$.

Case $m \in (-\infty, -2)$: Interior solution III

Function $r(m)$ decreases from $r = \text{constant}$ to $r = 0$. At the right border of interval asymptotics (4.51) is still valid. $f(m) > 0$ everywhere on interval, so the metrics has correct signature for $0 < r < r_{max}$, where $r_{max} = r(-\infty)$.

5 Another interior solution

In obtaining the solutions above it was assumed that m is a functions of r . One may also search for solutions with m to be constant. Assuming so, one obtains from (4.33) an equation for m :

$$\frac{\alpha_1(m) - 1}{\alpha_2(m) - 1} = m. \quad (5.1)$$

The numerical computation of (5.1) gives the root, $m_0 \approx -1.60808367$; respectively, $\alpha_1(m_0) \approx -10.8671667$. The solution for metrics, due to (4.31) and (4.32), is:

$$ds^2 = \left(\frac{r_0^2}{r^2} - 1\right) \left(\frac{r}{r_0}\right)^{2m_0^2} c^2 dt^2 - m_0^2 \left(1 - \frac{r^2}{r_0^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (5.2)$$

Here

$$r_0 = \frac{L}{2\sqrt{|\alpha_1(m_0)| + 1}}. \quad (5.3)$$

Solution (5.2) makes sense for $r \in [0, r_0]$.

6 Conclusion

Among static, spherically symmetric solutions for metrics, predicted by the Born–Infeld theory, the most interesting is the exterior one, having asymptotic of the Schwarzschild solution on $r \rightarrow \infty$. This solution exhibits at small radial distances behavior, dramatically different from those of the traditional solution for a black hole.

Opposite to a black hole solution with a spacetime singularity, (a timelike curve $r = 0$), one deals now with a hypersurface, $r = r_{min} \approx 0.333 \times (r_g L^2)^{\frac{1}{3}}$, which presents a boundary to the solution. For $r < r_{min}$ the solution (4.39), (4.40) cease to be valid, since in this region determinant of the metric tensor, calculated with (4.39), (4.40), $g > 0$. This circumstance doesn't exclude the region from the physical description. It means, instead, that one has to solve field equations (3.1), (3.2) (modified to include the matter), for solution corresponding to the interior of the mass, and sew both interior and exterior solutions on the boundary. One should emphasize, though, a remarkable fact, that the exterior solution “leaves a vacancy” (i.e. a spacial volume), $V \propto r_{min}^3$, for the mass. This

actually means that matter cannot be squeezed (even by forces other than gravity) beyond the radius r_{min} , otherwise the field equations wouldn't be consistent.² Using dimensional analysis, one obtains the average density of the mass, $\rho \propto M/r_{min}^3$; the coefficient of proportionality depends on geometry of space inside the volume. This density doesn't depend on the value of the mass, M , and, in fact, is of order of magnitude of the Planck's density. Thus, Born–Infeld gravity is more ‘benign’ than that of Einstein: it doesn't squeeze matter more than to the Planck's density (approximately). Unlike a black hole singularity, the boundary $r = r_{min}$ doesn't have physical infinities on it. Really, the curvature invariants, $R^{abcd}R_{abcd}$, and $R_{ab}^{cd}R_{eh}^{ab}R_{cd}^{eh}$ are finite on the boundary. Here R_{bcd}^a is the Riemann tensor, calculated with metrics (4.47) and (4.48). In short, Born–Infeld theory replaces a black hole's point-like spacial singularity with infinite density of mass, by a ‘ball of matter’ with finite density of order of magnitude of the Planck's density. The same solution, (4.39), (4.40) may serve as an exterior part in the case when the mass has lesser than ‘ultimate’ density; one should, then, sew solutions (both exterior and interior for the case) together at some $r > r_{min}$.

Another notable difference, is that surface of a horizon, which is defined in Einstein's theory by equation $r = r_g$, in Born–Infeld theory should be defined by equation $r = r_h$, where $r_h = r(m_h)$, and m_h is the root of equation $f(m_h) = 0$ (c.f. (4.49)). For example, for $r_g \gg r_*$, one obtains $r_h \approx r_g - r_*^6/r_g^5$, where r_* is given by (4.43).

For masses $M \gg 10^{-5}$ cm, spacetime geometry is similar to that described by the Schwarzschild solution in a sense that the horizon exists at $r \approx r_g \gg r_{min}$. As the mass, M , decreases, r_h decreases faster than $r_{min} \propto \sqrt[3]{r_g}$, so that both surfaces (the horizon and the boundary) fuse at $r \approx 0.185L$. Further decrease of the mass leads to disappearance of the horizon, so that the boundary becomes “naked”.

It makes sense to consider an interaction of two microscopic, ‘ultimately squeezed’ masses, gravitating with each other according to the classical potential (c.f. (4.41)). Take masses, $M \approx 10^{-20}$ g. Then, from (4.41) follows, that at sufficiently large distances, the potential, $V(r) \sim \frac{3r_*^6}{r^6} - \frac{r_g}{r}$. For this case, $r_g \sim 10^{-15}L$, and $r_* \sim (r_g/L)^{\frac{1}{3}}L \sim 10^{-5}L$. (From (4.43) follows, $r_* \approx 0.482(r_g L^2)^{\frac{1}{3}}$, so that $r_* > r_{min}$.) The distance, r_{eq} , at which potential energy $V(r)$ has minimum, $r_{eq} \sim (r_g/L)^{\frac{1}{5}}L \sim 10^{-3}L \sim 10^{-36}$ cm. Note, that $r_*/r_{eq} \sim 10^{-2}$, so using of formula (4.41) is justified. Thus, for $r > r_{eq}$, the mass attracts, and on distances $r < r_{eq}$, it repulses. This repulsion may prevent fusion of two (or more) ‘particles’ with sufficiently small masses, or at least make such fusion less probable. On the other hand, for masses with $r_g > 0.185L$, the horizon exists. One may guess, that such a mass, undergoing implosion and passing beyond the horizon, will be squeezed up to the boundary, i.e. will form a core with average density of order of magnitude of the Planck's density.

The interior solution, (5.2), is unique in a sense that it has fixed, well defined dimensions. This solution corresponds to a closed space with finite volume, $V = 2|m_0|\pi^2 r_0^3$. This microscopic universe might be considered as a candidate for a “seed”, which could

²One may guess, that the minimal radius, r_{min} , will change after ‘switching on’ other interactions; though, one expects that general structure of the solution will remain the same.

inflate under certain circumstances (Big Bang) into a Universe, similar to ours.

7 Acknowledgements

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